2.6.32. Uncle Harry and Aunt Minnie will both be attending your next family reunion. Unfortunately, they hate each other. Unless they are seated with at least two people between them, they are likely to get into a shouting match. The side of the table at which they will be seated has seven chairs. How many seating arrangements are available for those seven people if a safe distance is to be maintained between your aunt and your uncle?

2.6.33. In how many ways can the digits 1 through 9 be arranged such that

(a) all the even digits precede all the odd digits?
(b) all the even digits are adjacent to each other?
(c) two even digits begin the sequence and two even digits end the sequence?
(d) the even digits appear in either ascending or descending order?

Counting Permutations (when the objects are not all distinct)

The corollary to Theorem 2.6.1 gives a formula for the number of ways an entire set of n objects can be permuted *if the objects are all distinct*. Fewer than $n!$ permutations are possible, though, if some of the objects are identical. For example, there are $3! = 6$ ways to permute the three distinct objects A, B, and C:

ABC
ACB
BAC
BCA
CAB
CBA

If the three objects to permute, though, are A, A, and B—that is, if two of the three are identical—the number of permutations decreases to three:

AAB
ABA
BAA

As we will see, there are many real-world applications where the n objects to be permuted belong to r different categories, each category containing one or more identical objects.

Theorem 2.6.2 *The number of ways to arrange n objects, n_1 being of one kind, n_2 of a second kind, ..., and n_r of an rth kind, is*

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

where $\sum_{i=1}^{r} n_i = n$.

Proof Let N denote the total number of such arrangements. For any one of those N, the similar objects (if they were actually different) could be arranged in $n_1!\,n_2!\cdots n_r!$ ways. (Why?) It follows that $N \cdot n_1!\,n_2!\cdots n_r!$ is the total number of ways to arrange n (distinct) objects. But $n!$ equals that same number. Setting $N \cdot n_1!\,n_2!\cdots n_r!$ equal to $n!$ gives the result. □

Comment Ratios like $n!/(n_1!n_2!\cdots n_r!)$ are called *multinomial coefficients* because the general term in the expansion of

$$(x_1 + x_2 + \cdots + x_r)^n$$

is

$$\frac{n!}{n_1!n_2!\cdots n_r!} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

Example 2.6.14 A pastry in a vending machine costs 85¢. In how many ways can a customer put in two quarters, three dimes, and one nickel?

Figure 2.6.13

If all coins of a given value are considered identical, then a typical deposit sequence, say, *QDDQND* (see Figure 2.6.13), can be thought of as a permutation of $n = 6$ objects belonging to $r = 3$ categories, where

$$n_1 = \text{number of nickels} = 1$$

$$n_2 = \text{number of dimes} = 3$$

$$n_3 = \text{number of quarters} = 2$$

By Theorem 2.6.2, there are sixty such sequences:

$$\frac{n!}{n_1!n_2!n_3!} = \frac{6!}{1!3!2!} = 60$$

Of course, had we assumed the coins were distinct (having been minted at different places and different times), the number of distinct permutations would have been 6!, or 720. ■

Example 2.6.15 Prior to the seventeenth century there were no scientific journals, a state of affairs that made it difficult for researchers to document discoveries. If a scientist sent a copy of his work to a colleague, there was always a risk that the colleague might claim it as his own. The obvious alternative—wait to get enough material to publish a book—invariably resulted in lengthy delays. So, as a sort of interim documentation, scientists would sometimes send each other anagrams—letter puzzles that, when properly unscrambled, summarized in a sentence or two what had been discovered.

When Christiaan Huygens (1629–1695) looked through his telescope and saw the ring around Saturn, he composed the following anagram (191):

$$aaaaaaa, ccccc, d, eeeee, g, h, iiiiiii, llll, mm,$$

$$nnnnnnnnn, oooo, pp, q, rr, s, ttttt, uuuuu$$

How many ways can the sixty-two letters in Huygens's anagram be arranged?

Let $n_1(=7)$ denote the number of a's, $n_2(=5)$ the number of c's, and so on. Substituting into the appropriate multinomial coefficient, we find

$$N = \frac{62!}{7!5!1!5!1!1!7!4!2!9!4!2!1!2!1!5!5!}$$

as the total number of arrangements. To get a feeling for the magnitude of N, we need to apply Stirling's formula to the numerator. Since

$$62! \doteq \sqrt{2\pi} e^{-62} 62^{62.5}$$

then

$$\begin{aligned}\log(62!) &\doteq \log\left(\sqrt{2\pi}\right) - 62 \cdot \log(e) + 62.5 \cdot \log(62)\\ &\doteq 85.49731\end{aligned}$$

The antilog of 85.49731 is 3.143×10^{85}, so

$$N \doteq \frac{3.143 \times 10^{85}}{7!5!1!5!1!1!7!4!2!9!4!2!1!2!1!5!5!}$$

is a number on the order of 3.6×10^{60}. Huygens was clearly taking no chances! (*Note:* When appropriately rearranged, the anagram becomes "Annulo cingitur tenui, plano, nusquam cohaerente, ad eclipticam inclinato," which translates to "Surrounded by a thin ring, flat, suspended nowhere, inclined to the ecliptic.") ■

Example 2.6.16 What is the coefficient of x^{23} in the expansion of $(1 + x^5 + x^9)^{100}$?

To understand how this question relates to permutations, consider the simpler problem of expanding $(a + b)^2$:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a \cdot a + a \cdot b + b \cdot a + b \cdot b\\ &= a^2 + 2ab + b^2\end{aligned}$$

Notice that each term in the first $(a + b)$ is multiplied by each term in the second $(a + b)$. Moreover, the coefficient that appears in front of each term in the expansion corresponds to the number of ways that that term can be formed. For example, the 2 in the term $2ab$ reflects the fact that the product ab can result from two different multiplications:

$$\underbrace{(a+b)(a+b)}_{ab} \qquad \text{or} \qquad (a+\underbrace{b)\,(a}_{ab}+b)$$

By analogy, the coefficient of x^{23} in the expansion of $(1 + x^5 + x^9)^{100}$ will be the number of ways that one term from each of the one hundred factors $(1 + x^5 + x^9)$ can be multiplied together to form x^{23}. The only factors that will produce x^{23}, though, are the set of two x^9's, one x^5, and ninety-seven 1's:

$$x^{23} = x^9 \cdot x^9 \cdot x^5 \cdot 1 \cdot 1 \cdots 1$$

It follows that the *coefficient* of x^{23} is the number of ways to permute two x^9's, one x^5, and ninety-seven 1's. So, from Theorem 2.6.2,

$$\text{coefficient of } x^{23} = \frac{100!}{2!1!97!}$$
$$= 485,100$$

■

Example 2.6.17 A palindrome is a phrase whose letters are in the same order whether they are read backward or forward, such as Napoleon's lament

Able was I ere I saw Elba.

or the often-cited

Madam, I'm Adam.

Words themselves can become the units in a palindrome, as in the sentence

Girl, bathing on Bikini, eyeing boy,
finds boy eyeing bikini on bathing girl.

Suppose the members of a set consisting of four objects of one type, six of a second type, and two of a third type are to be lined up in a row. How many of those permutations are palindromes?

Think of the twelve objects to arrange as being four A's, six B's, and two C's. If the arrangement is to be a palindrome, then half of the A's, half of the B's, and half of the C's must occupy the first six positions in the permutation. Moreover, the final six members of the sequence must be in the reverse order of the first six. For example, if the objects comprising the first half of the permutation were

$$C \quad A \quad B \quad A \quad B \quad B$$

then the last six would need to be in the order

$$B \quad B \quad A \quad B \quad A \quad C$$

It follows that the number of palindromes is the number of ways to permute the first six objects in the sequence, because once the first six are positioned, there is only one arrangement of the last six that will complete the palindrome. By Theorem 2.6.2, then,

$$\text{number of palindromes} = 6!/(2!3!1!) = 60$$

■

Example 2.6.18 A deliveryman is currently at Point X and needs to stop at Point 0 before driving through to Point Y (see Figure 2.6.14). How many different routes can he take without ever going out of his way?

Notice that any admissible path from, say, X to 0 is an ordered sequence of 11 "moves"—nine east and two north. Pictured in Figure 2.6.14, for example, is the particular X to 0 route

$$E \quad E \quad N \quad E \quad E \quad E \quad E \quad N \quad E \quad E \quad E$$

Similarly, any acceptable path from 0 to Y will necessarily consist of five moves east and three moves north (the one indicated is $E\ E\ N\ N\ E\ N\ E\ E$).

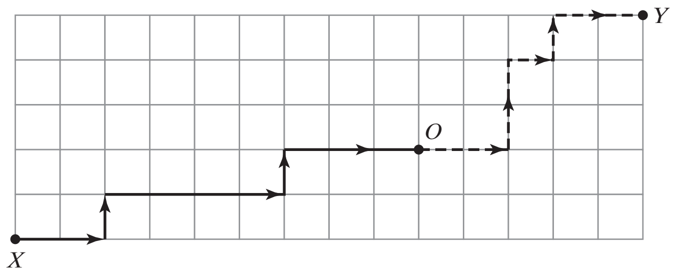


Figure 2.6.14

Since each path from X to 0 corresponds to a unique permutation of nine E's and two N's, the *number* of such paths (from Theorem 2.6.2) is the quotient

$$11!/(9!2!) = 55$$

For the same reasons, the number of different paths from 0 to Y is

$$8!/(5!3!) = 56$$

By the multiplication rule, then, the total number of admissible routes from X to Y that pass through 0 is the product of 55 and *56*, or *3080*. ■

Example 2.6.19

A burglar is trying to deactivate an alarm system that has a six-digit entry code. He notices that three of the keyboard buttons—the 3, the 4, and the 9—are more polished than the other seven, suggesting that only those three numbers appear in the correct entry code. Trial and error may be a feasible strategy, but earlier misadventures have convinced him that if his probability of guessing the correct code in the first thirty minutes is not at least 70%, the risk of getting caught is too great. Given that he can try a different permutation every five seconds, what should he do? He could look for an unlocked window to crawl through (or, here's a thought, get an honest job!). Deactivating the alarm, though, is not a good option.

Table 2.6.3 shows that 570 six-digit permutations can be made from the numbers 3, 4, and 9.

Table 2.6.3

Form of Permutations	Example	Number
One digit appears four times; other digits appear once	449434	$6!/(4!1!1!) \times 3 = 90$
One digit appears three times; another appears twice; and a third appears once	944334	$6!/(3!2!1!) \times 3! = 360$
Each digit appears twice	439934	$6!/(2!2!2!) \times 1 = 120$
		TOTAL: 570

Guessing at the rate of one permutation every five seconds would allow 360 permutations to be tested in thirty minutes, but 360 is only 63% of 570, so the burglar's 70% probability criteria of success would not be met. (*Question*: The first factors in Column 3 of Table 2.6.3 are applications of Theorem 2.6.2 to the sample permutations shown in Column 2. What do the second factors in Column 3 represent?) ■

Questions

2.6.34. Which state name can generate more permutations, TENNESSEE or FLORIDA?

2.6.35. How many numbers greater than four million can be formed from the digits 2, 3, 4, 4, 5, 5, 5?

2.6.36. An interior decorator is trying to arrange a shelf containing eight books, three with red covers, three with blue covers, and two with brown covers.

(a) Assuming the titles and the sizes of the books are irrelevant, in how many ways can she arrange the eight books?
(b) In how many ways could the books be arranged if they were all considered distinct?
(c) In how many ways could the books be arranged if the red books were considered indistinguishable, but the other five were considered distinct?

2.6.37. Four Nigerians (A, B, C, D), three Chinese (#, *, &), and three Greeks (α, β, γ) are lined up at the box office, waiting to buy tickets for the World's Fair.

(a) How many ways can they position themselves if the Nigerians are to hold the first four places in line; the Chinese, the next three; and the Greeks, the last three?
(b) How many arrangements are possible if members of the same nationality must stay together?
(c) How many different queues can be formed?
(d) Suppose a vacationing Martian strolls by and wants to photograph the ten for her scrapbook. A bit myopic, the Martian is quite capable of discerning the more obvious differences in human anatomy but is unable to distinguish one Nigerian (N) from another, one Chinese (C) from another, or one Greek (G) from another. Instead of perceiving a line to be $B*\beta AD\#\&C\alpha\gamma$, for example, she would see *NCGNNCCNGG*. From the Martian's perspective, in how many different ways can the ten funny-looking Earthlings line themselves up?

2.6.38. How many ways can the letters in the word

SLUMGULLION

be arranged so that the three L's precede all the other consonants?

2.6.39. A tennis tournament has a field of $2n$ entrants, all of whom need to be scheduled to play in the first round. How many different pairings are possible?

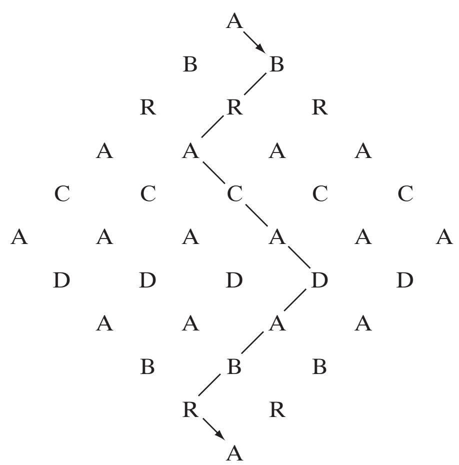
2.6.40. What is the coefficient of x^{12} in the expansion of $(1+x^3+x^6)^{18}$?

2.6.41. In how many ways can the letters of the word

ELEEMOSYNARY

be arranged so that the S is always immediately followed by a Y?

2.6.42. In how many ways can the word *ABRACADABRA* be formed in the array pictured below? Assume that the word must begin with the top A and progress diagonally downward to the bottom A.



2.6.43. Suppose a pitcher faces a batter who never swings. For how many different ball/strike sequences will the batter be called out on the fifth pitch?

2.6.44. What is the coefficient of $w^2x^3yz^3$ in the expansion of $(w+x+y+z)^9$?

2.6.45. Imagine six points in a plane, no three of which lie on a straight line. In how many ways can the six points be used as vertices to form two triangles? (*Hint:* Number the points 1 through 6. Call one of the triangles A and the other B. What does the permutation